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Separation in a two-dimensional Stokes flow inside an elliptic cylinder

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Abstract. The two-dimensional Stokes flow due to a line rotlet inside a fixed elliptic cylinder is investigated, where it is assumed that the line rotlet intersects the major axis of each elliptical cross-section of the cylinder. For the case in which the line rotlet coincides with the centre-line of the elliptic cylinder, it is shown that the number of eddies in the flow increases in a roughly linear way with the ratio of length to width of a cross-section of the cylinder. Moreover, results obtained by varying the rotlet position for several different fixed boundary shapes suggest that the aforementioned ratio, and not the rotlet position, is the principal determinant of the number of eddies.

1. Introduction

A well-known Stokes flow problem in which separation may occur is the two-dimensional flow in the annular region between two circular cylinders with parallel axes, resulting from the rotation of one or both of the cylinders. Jeffery [6], using a bipolar coordinate system, was one of the first to solve this problem; however, Jeffery was mainly interested in calculating the torques on the cylinders and did not discuss the flow structure. Wannier [9] was the first to discover that, for sufficiently large distances between the axes of the cylinders, an eddy is attached to the outer cylinder when that cylinder is not rotating. Ballal and Rivlin [1], whose analysis of the flow structure is the most comprehensive to date, have noticed that an eddy may be attached to the inner cylinder when that cylinder is not rotating and that two free eddies may exist when the cylinders rotate in the same direction. Ranger [8] has considered a similar flow in which the inner cylinder is replaced by a line rotlet. (A line rotlet may be regarded as a rotating circular cylinder of infinitesimal radius.) Ranger's flow exhibits all of the types of separation observed by Ballal and Rivlin, except those that occur when the inner cylinder is not rotating, and has the advantage of being much easier to describe mathematically than the flow between two cylinders.

Another example of Stokes flow separation is that of Moffatt [7] who showed that an infinite set of eddies exists in the Stokes flow induced by an arbitrary two-dimensional disturbance between parallel planar walls. Hackborn [5] examined the special case in which the disturbance driving this flow is a line rotlet.

In this paper, we will investigate the two-dimensional Stokes flow due to a line rotlet inside a fixed elliptic cylinder. This flow provides an important link between the flow due to a line rotlet inside a fixed circular cylinder, in which at most one eddy can exist, and the flow due to a line rotlet between fixed parallel planes, in which an infinite set of eddies exists. Moreover, the flow features described in this paper are expected to be present in the Stokes flow induced by a rotating circular cylinder inside a fixed elliptic cylinder.

It is demonstrated here that, when the line rotlet coincides with the centre-line of the elliptic cylinder, the number of eddies in the flow increases in an approximately linear way with the ratio of length to width of the elliptical cross-section of the cylinder. Furthermore, our results suggest that this ratio is much more important in determining the number of eddies than the position of the rotlet on the major axis of the cross-section.

2. Statement of the problem

The flow to be studied occurs within a fixed elliptic cylinder containing a homogeneous incompressible fluid of viscosity μ. A cross-section of the flow region is assumed to lie inside an ellipse whose major and minor axes coincide with the x- and y-axes respectively of an (x, y) plane, where x and y are Cartesian coordinates taken to be dimensionless relative to the length of the semi-minor axis of the boundary ellipse. Hence, the y-intercepts of this ellipse occur at $y = \pm 1$, while the x-intecepts occur at $x = \pm M$, where $M \geq 1$. It is assumed that the flow is driven by a line rotlet, of strength σ, coinciding with the line $x = c$, $y = 0$, where $0 \leq c < M$, and that the flow direction near the rotlet is counter-clockwise. (By definition, a line rotlet is a singularity which exerts a torque per unit length of magnitude $4\pi\sigma\mu$ on the surrounding fluid. Mathematically, a line rotlet is equivalent to a line vortex of potential flow theory. However, in Stokes flow problems, the term 'rotlet' is often used to denote both it and its three-dimensional analogue. (See Blake and Chwang [2], Hackborn [5], and Ranger [8] for example.) A cross-section of the flow region is shown in Fig. 1.

It is convenient to introduce coordinates (ξ, η) such that the boundary ellipse in the (x, y) plane coincides with the curve $\xi = \alpha$, where α is a positive constant. These coordinates are related conformally to (x, y) by

$$x = k \cosh \xi \cos \eta , \qquad y = k \sinh \xi \sin \eta , \tag{2.1}$$

where $k = \operatorname{csch} \alpha$ is a dimensionless scaling constant. With k as given, the length (with respect to x and y) of the semi-minor axis of the ellipse $\xi = \alpha$ is unity, as required, and the length of the semi-major axis is $M = \coth \alpha$. Also note that a cross-section of the flow region is defined by $0 \leq \xi < \alpha$, $-\pi < \eta \leq \pi$.

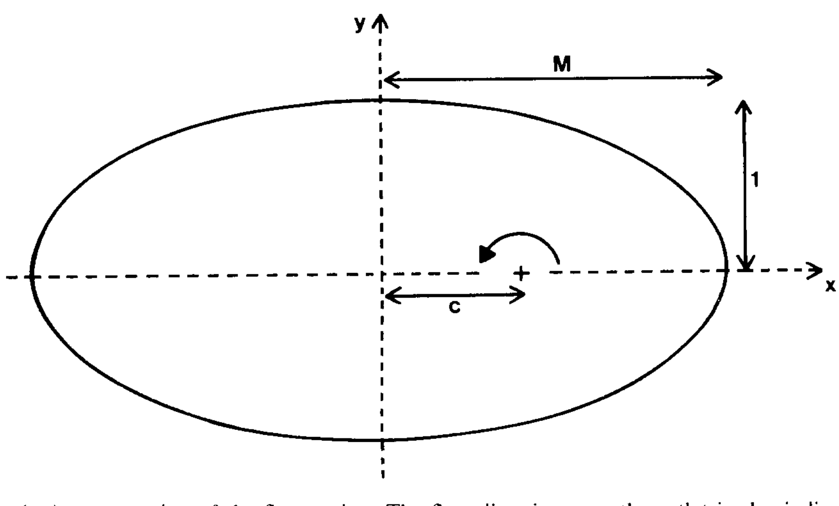

Fig. 1. A cross-section of the flow region. The flow direction near the rotlet is also indicated.

The problem is best expressed using a stream function ψ, assumed to be dimensionless relative to σ. The components u and v of the dimensionless fluid velocity in the x- and y-directions respectively are then given by

$$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}. \tag{2.2}$$

The stream function for a Stokes flow satisfies the biharmonic equation

$$\nabla^4 \psi = 0, \qquad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{2.3}$$

Using (2.2), the no-slip boundary condition on the ellipse $\xi = \alpha$ may be expressed by

$$\psi = \frac{\partial \psi}{\partial \xi} = 0, \quad \text{at } \xi = \alpha. \tag{2.4}$$

Finally, the presence of the line rotlet at $x = c$, $y = 0$ requires that

$$\psi = \log R + \tilde{\psi}, \qquad R = [(x-c)^2 + y^2]^{1/2}, \tag{2.5}$$

where $\tilde{\psi}$ describes a Stokes flow with no singularities in the flow region.

3. Solution of the problem

It is helpful to let

$$c = k \cosh \beta, \tag{3.1}$$

where $\beta = \mathrm{i}\zeta$, with $0 < \zeta \leq \pi/2$, for $0 \leq c < k$, and $0 \leq \beta < \alpha$, for $k \leq c < M$. By combining (2.1) and (3.1) with the expression for R in (2.5) and manipulating the result, we obtain

$$R = k[(\cosh \xi \cosh \beta - \cos \eta)^2 - \sinh^2\xi \sinh^2\beta]^{1/2}, \tag{3.2}$$

and it follows from (3.2) that

$$\log R = \log k + \tfrac{1}{2}\log[\cosh(\xi + \beta) - \cos \eta] + \tfrac{1}{2}\log[\cosh(\xi - \beta) - \cos \eta]. \tag{3.3}$$

Now, using Maclaurin expansions, it is found that

$$\begin{aligned} \log\left(\frac{1+t^2}{2t} - \cos \eta\right) &= -\log 2t + \log(1 - t\,\mathrm{e}^{\mathrm{i}\eta}) + \log(1 - t\,\mathrm{e}^{-\mathrm{i}\eta}) \\ &= -\log 2t - 2\sum_{n=1}^{\infty} n^{-1} t^n \cos n\eta, \end{aligned} \tag{3.4}$$

for $|t| < 1$. Letting $t = \mathrm{e}^{-(\xi \pm \beta)}$ in (3.4) gives

$$\log[\cosh(\xi \pm \beta) - \cos \eta] = \xi \pm \beta - \log 2 - 2 \sum_{n=1}^{\infty} n^{-1} e^{-n(\xi \pm \beta)} \cos n\eta , \tag{3.5}$$

for $\xi > \mathrm{Re}(\beta)$. Hence, from (3.3), (3.5) and the fact that $k = \operatorname{csch} \alpha$,

$$\log R = \xi - \log(2 \sinh \alpha) - 2 \sum_{n=1}^{\infty} n^{-1} e^{-n\xi} \cosh n\beta \cos n\eta , \tag{3.6}$$

provided that $\xi > \mathrm{Re}(\beta)$.

Noting that if ϕ is harmonic with respect to x and y then both ϕ and $x\phi$ are biharmonic, a suitable representation for the function $\tilde{\psi}$ used in (2.5) is

$$\tilde{\psi} = \phi_1 + x\phi_2 , \tag{3.7}$$

where ϕ_1 and ϕ_2 are linear combinations of the harmonics $\cosh n\xi \cos n\eta$, for $n = 0, 1, 2, \ldots$. This representation for $\tilde{\psi}$ is an even 2π-periodic function of η, as required by the symmetry of the stated problem. Furthermore, it is easily shown that the Stokes flow associated with this $\tilde{\psi}$ has no singularities in the flow region, not even at $(x, y) = (\pm k, 0)$ (the foci of the ellipse $\xi = \alpha$) where the inverse of transformation (2.1) is singular. On combining (2.5) with an expanded version of (3.7), we find that

$$\begin{aligned}\psi = {} & \log R + A_0 + B_1 \cosh 2\xi + (A_1 \cosh \xi + B_2 \cosh 3\xi) \cos \eta \\ & + \sum_{n=2}^{\infty} [A_n \cosh n\xi + B_{n+1} \cosh(n+2)\xi + B_{n-1} \cosh(n-2)\xi] \cos n\eta ,\end{aligned} \tag{3.8}$$

where A_n $(n = 0, 1, 2, \ldots)$ and B_n $(n = 1, 2, 3, \ldots)$ are constants to be determined.

From boundary condition (2.4) and expansions (3.6) and (3.8), it is readily seen that the constants A_n and B_n satisfy the following equations:

$$A_0 + B_1 \cosh 2\alpha = \log(2 \sinh \alpha) - \alpha , \tag{3.9}$$

$$2B_1 \sinh 2\alpha = -1 , \tag{3.10}$$

$$A_1 \cosh \alpha + B_2 \cosh 3\alpha = 2 e^{-\alpha} \cosh \beta , \tag{3.11}$$

$$A_1 \sinh \alpha + 3B_2 \sinh 3\alpha = -2 e^{-\alpha} \cosh \beta , \tag{3.12}$$

$$A_n \cosh n\alpha + B_{n+1} \cosh(n+2)\alpha + B_{n-1} \cosh(n-2)\alpha = 2n^{-1} e^{-n\alpha} \cosh n\beta , \tag{3.13}$$

$$\begin{aligned}& nA_n \sinh n\alpha + (n+2)B_{n+1} \sinh(n+2)\alpha + (n-2)B_{n-1} \sinh(n-2)\alpha \\ & \quad = -2 e^{-n\alpha} \cosh n\beta ,\end{aligned} \tag{3.14}$$

where $n = 2, 3, 4, \ldots$ in (3.13) and (3.14). Eliminating A_n $(n = 0, 1, 2, \ldots)$ in equations (3.9) to (3.14) and defining

$$C_n = (\sinh 2n\alpha + n \sinh 2\alpha) B_n , \tag{3.15}$$

for $n = 1, 2, 3, \ldots$, produces the recursion

$$C_1 = -1\,, \qquad C_2 = -2\cosh\beta\,, \qquad C_{n+1} - C_{n-1} = -2\cosh n\beta\,, \tag{3.16}$$

for $n = 2, 3, 4, \ldots$. The solution to (3.16) is found to be

$$C_n = \frac{-\sinh n\beta}{\sinh\beta}\,, \tag{3.17}$$

for $n = 1, 2, 3, \ldots$. (If $\beta = 0$, the limit of the expression in (3.17) as β approaches zero should be understood.) Thus, from (3.15) and (3.17),

$$B_n = \frac{-\sinh n\beta}{(\sinh 2n\alpha + n\sinh 2\alpha)\sinh\beta}\,, \tag{3.18}$$

for $n = 1, 2, 3, \ldots$. Hence, the required solution for the stream function ψ is given by (3.8), where B_n is given by (3.18) and A_n $(n = 0, 1, 2, \ldots)$ may be obtained directly by substituting the values of B_n into (3.9), (3.11) and (3.13).

It is also useful to find an expression for the scalar vorticity $\nabla^2\psi$ since its sign at a boundary point indicates the flow direction near that point. From (2.1) and (3.8), we obtain

$$\nabla^2\psi = \frac{4\sinh^2\alpha}{\sinh^2\xi + \sin^2\eta}\sum_{n=1}^{\infty} nB_n[\cosh(n+1)\xi\cos(n-1)\eta - \cosh(n-1)\xi\cos(n+1)\eta]\,. \tag{3.19}$$

4. Flow description

We will begin our flow description by considering the special case $c = 0$, that is, the case in which the line rotlet coincides with $x = y = 0$, the centre of the ellipse $\xi = \alpha$.

Recall that the length (with respect to x and y) of the semi-minor axis of the boundary ellipse $\xi = \alpha$ is fixed at unity, while the length of the semi-major axis is $M = \coth\alpha$. Therefore, the ellipse $\xi = \alpha$ becomes a circle of radius unity in the limit as $\alpha \to \infty$; hence, when $c = 0$, there is no separation for sufficiently large α since the Stokes flow due to a line rotlet at the centre of a fixed circular cylinder (see Ranger [8]) exhibits no separation. However, the ellipse $\xi = \alpha$ approximates parallel planes at $y = \pm 1$ increasingly well as $\alpha \to 0^+$; thus, we expect the number of eddies to increase without limit as $\alpha \to 0^+$ since the Stokes flow due to a line rotlet between fixed parallel planes (see Hackborn [5]) has an infinite set of eddies.

In the following analysis of the case $c = 0$, it is assumed that the eddies created as $\alpha \to 0^+$ emerge at the vertices (i.e. the endpoints of the major axis) of the ellipse $\xi = \alpha$. (This is a reasonable assumption that is amply supported by extensive numerical computation of the vorticity on the boundary.) The relationship between α and the number of eddies is explored using the quantity $W(\alpha)$ which denotes the value of $\nabla^2\psi$ at the vertex $(\xi, \eta) = (\alpha, 0)$ when $c = 0$. Noting that $\beta = i\pi/2$ when $c = 0$ from (3.1), equations (3.18) and (3.19) imply that

$$W(\alpha) = 8\sinh\alpha\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)\sinh(2n-1)\alpha}{\sinh(4n-2)\alpha + (2n-1)\sinh 2\alpha}\,. \tag{4.1}$$

The behaviour of $W(\alpha)$ as $\alpha \to 0^+$ will be exposed using a method employed by Davis et al. in [4]. First, we define the function

$$f(z) = \frac{z \sinh z\alpha \sec(\pi z/2)}{\sinh 2z\alpha + z \sinh 2\alpha}, \tag{4.2}$$

where z is a complex variable. Let Γ be the contour in the complex z plane consisting of the infinite semi-circle S in the half-plane $\mathrm{Re}(z) > 0$ together with the imaginary axis I and having a positive orientation. Due to the behaviour of $f(z)$ at infinity, $\int_S f(z)\,\mathrm{d}z = 0$. Also, $\int_I f(z)\,\mathrm{d}z = 0$ since $f(z)$ has no singularities on the imaginary axis (except a removable singularity at $z = 0$) and $f(it)$ is an odd function of the real variable t. Hence, $\int_\Gamma f(z)\,\mathrm{d}z = 0$, and so the sum of the residues of $f(z)$ at its singularities in the half-plane $\mathrm{Re}(z) > 0$ must vanish. Now, the only singularities of $f(z)$ in the half-plane $\mathrm{Re}(z) > 0$ are simple poles occurring at $z = 2n - 1$, $-\mathrm{i}\gamma_n/2\alpha$, $\mathrm{i}\bar{\gamma}_n/2\alpha$, for $n = 1, 2, 3, \ldots$, where the constants γ_n are the values in the first quadrant satisfying

$$2\alpha \sin \gamma_n + \gamma_n \sinh 2\alpha = 0, \tag{4.3}$$

and are ordered by increasing real part. Furthermore, it follows from (4.1) and (4.2) that $W(\alpha)$ is simply the product of $4\pi \sinh \alpha$ and the sum of the residues of $f(z)$ at $z = 2n - 1$ $(n = 1, 2, 3, \ldots)$; hence, $W(\alpha)$ can be written in terms of the sum of the residues of $f(z)$ at $z = -\mathrm{i}\gamma_n/2\alpha$, $\mathrm{i}\bar{\gamma}_n/2\alpha$ $(n = 1, 2, 3, \ldots)$, giving

$$W(\alpha) = \frac{4\pi \sinh \alpha}{\alpha} \mathrm{Re} \sum_{n=1}^{\infty} \frac{\gamma_n \sin(\gamma_n/2) \operatorname{sech}(\pi\gamma_n/4\alpha)}{2\alpha \cos \gamma_n + \sinh 2\alpha}. \tag{4.4}$$

Now, it can be shown from (4.3) that

$$\gamma_n = \lambda_n + \mathrm{O}(\alpha^2), \quad \text{as } \alpha \to 0^+, \tag{4.5}$$

for $n = 1, 2, 3, \ldots$, where the constants λ_n are the values in the first quadrant satisfying $\sin \lambda_n = -\lambda_n$ and are ordered by increasing real part. (The notation for the constants λ_n comes from Buchwald [3] who used them in a study of plane elastostatics and tabulated the first five values. They also arise in Davis et al. [4], Hackborn [5] and other articles on Stokes flow.) Substitution of (4.5) into (4.4) yields

$$W(\alpha) = \frac{4\pi}{\alpha} \mathrm{Re} \sum_{n=1}^{\infty} \frac{\lambda_n \sin(\lambda_n/2) \exp(-\pi\lambda_n/4\alpha)(1 + \mathrm{O}(\alpha))}{\cos \lambda_n + 1}, \tag{4.6}$$

as $\alpha \to 0^+$. The second and subsequent terms of the series in (4.6) may be absorbed into the $\mathrm{O}(\alpha)$ part of the first term. Carrying this out and rewriting the first term gives

$$W(\alpha) = \frac{4\pi|\Lambda|}{\alpha} (1 + \mathrm{O}(\alpha)) \exp\left(\frac{-\pi a}{4\alpha}\right) \cos\left(\frac{\pi b}{4\alpha} - \arg(\Lambda) + \mathrm{O}(\alpha)\right), \tag{4.7}$$

as $\alpha \to 0^+$, where

$$a = \mathrm{Re}(\lambda_1), \qquad b = \mathrm{Im}(\lambda_1), \qquad \Lambda = \frac{\lambda_1 \sin(\lambda_1/2)}{\cos \lambda_1 + 1}. \tag{4.8}$$

To seven significant digits, $a \approx 4.212392$, $b \approx 2.250729$ and $\arg(\Lambda) \approx -1.833636$. Changes in the sign of $W(\alpha)$ occur when the argument of the cosine factor in (4.7) equals $(n + 1/2)\pi$ for some integer n, and, ignoring the $O(\alpha)$ term, this condition yields a positive solution for α if and only if $n = 1, 2, 3, \ldots$. This suggests (and numerical results confirm) that

$$\frac{\pi b}{4\alpha_n} = \left(n + \frac{1}{2}\right)\pi + \arg(\Lambda) + O(\alpha_n)\,, \tag{4.9}$$

as $n \to \infty$, where α_n $(n = 1, 2, 3, \ldots)$ are the values of α, in decreasing order, at which $W(\alpha)$ changes sign. Consequently, using (4.9) and the fact that $\coth \alpha = \alpha^{-1} + O(\alpha)$ for small α, we have

$$M_n = \frac{4}{\pi b}\left[\left(n + \frac{1}{2}\right)\pi + \arg(\Lambda)\right] + O(n^{-1})\,, \tag{4.10}$$

as $n \to \infty$, where $M_n = \coth \alpha_n$ is the length of the semi-major axis of the ellipse $\xi = \alpha_n$.

If n is small, M_n can be computed numerically using the expression for $W(\alpha)$ given in (4.1). The first four values of M_n have been computed in this way and are provided, correct to five significant digits, in Table 1 alongside the corresponding approximations to M_n obtained from (4.10). Evidently, the approximations to M_n are in good agreement with the exact values; for $n \geq 2$, the approximations are correct to at least three signficant digits, virtually exact for most practical purposes.

For the case $c = 0$, a new eddy emerges at each vertex (i.e. at the intercepts $x = \pm M$) of the ellipse $\xi = \alpha$ as M increases through M_n $(n = 1, 2, 3, \ldots)$, or, equivalently, as α decreases through α_n. Therefore, if $1 \leq M \leq M_1$, there are no eddies in the flow when $c = 0$; however, if $M_n < M \leq M_{n+1}$, there are $2n$ eddies, n eddies on each side of a 'separation region' (not considered to be an eddy) containing the rotlet. Some streamlines of flows for which $c = 0$ are depicted in Fig. 2(a) (for $M = 2$) and Fig. 2(b) (for $M = 4$). The dividing streamlines, on which $\psi = 0$, shown in Fig. 2 were accurately calculated using (3.8) to determine points on these streamlines in the interior of the flow region and (3.19) to determine the points at which they intersect the boundary. It is noteworthy that there is a roughly linear relationship between M and the number of eddies when $c = 0$, since (4.10) implies that $M_{n+1} - M_n$ approaches $4/b \approx 1.7772$ as $n \to \infty$ (and $M_{n+1} - M_n$ is close to $4/b$ even when n is small). Therefore, whenever M is increased by $4/b$, the number of eddies is likely to increase by 2.

The changes induced in the flow by varying c while holding M constant will now be examined for several values of M. The main tool in this investigation is the boundary vorticity, computed using (3.19) with $\xi = \alpha$.

The flow for the case $M = 1$ (i.e. the case of a line rotlet inside a circular cylinder) has been studied previously by Ranger [8], who found that there are no eddies in this flow if

Table 1. Comparison of exact values of M_n obtained from (4.1) with approximate values obtained from (4.10)

n	Exact M_n	Approximate M_n
1	1.6439	1.6285
2	3.4098	3.4057
3	5.1853	5.1829
4	6.9618	6.9601

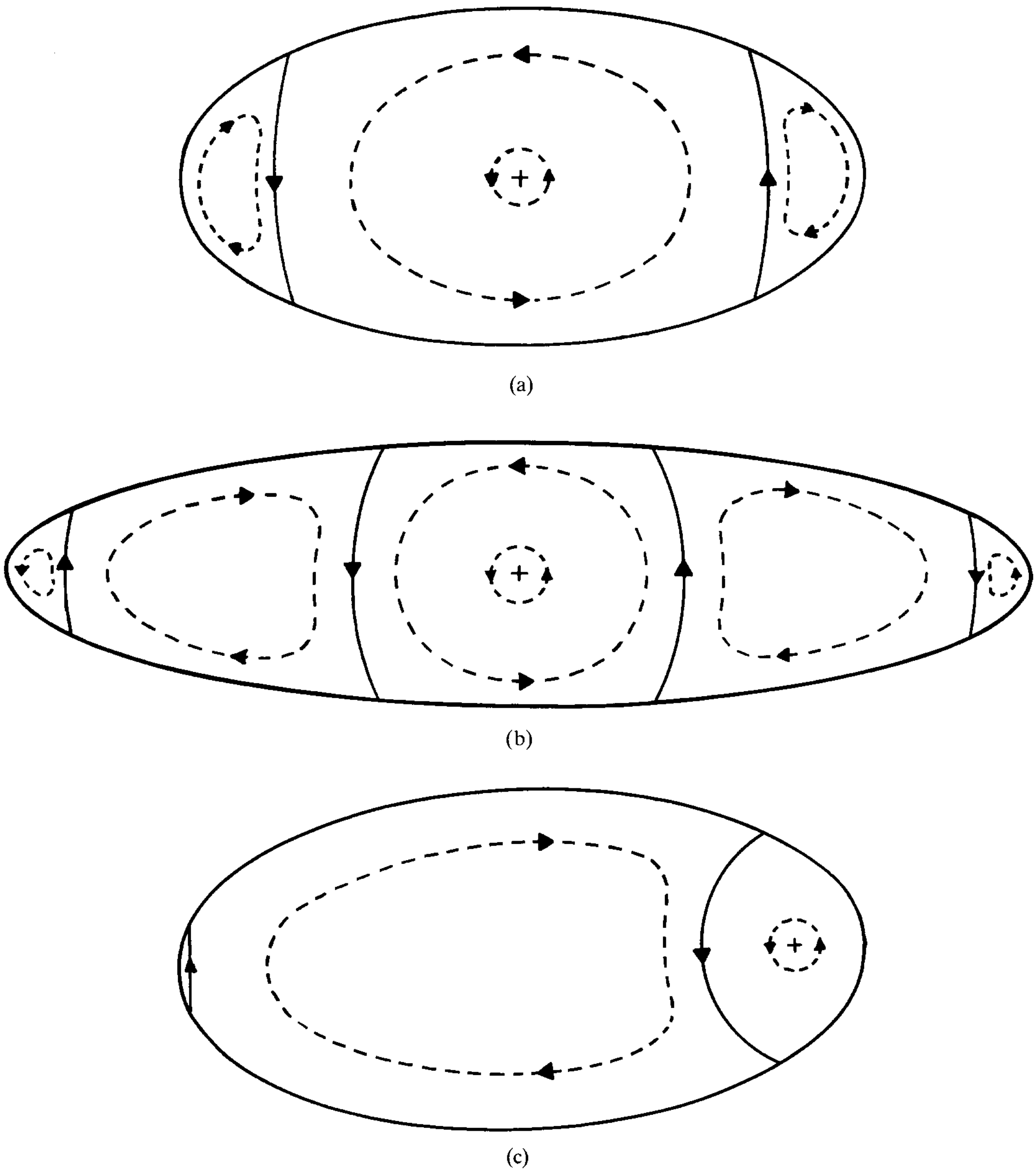

Fig. 2. Streamlines of the flow for (a) $M = 2$, $c = 0$, (b) $M = 4$, $c = 0$, and (c) $M = 2$, $c = 1.6$. The solid streamlines are exact plots; the dashed streamlines are schematic. The rotlet position is indicated by '+'.

$c \leqslant \sqrt{2} - 1$, but that a single eddy does exist if $c > \sqrt{2} - 1$. This eddy first emerges at the intercept $x = -1$ of the circle as c increases.

When $M = 2$ and $c = 0$, there are two eddies in the flow as shown in Fig. 2(a). As c increases from 0 while M remains at 2, the eddy adjacent to the vertex at $x = 2$ shrinks and eventually vanishes when $c \approx 0.4873$; as c continues to increase, a new eddy eventually emerges at the vertex at $x = -2$ when $c \approx 1.567$. Thus, when $M = 2$, there are either one or two eddies, depending on the value of c. Some streamlines of the flow for the case in which $M = 2$ and $c = 1.6$ are shown in Fig. 2(c).

When $M = 3$ and $c = 0$, there are again two eddies in the flow, and, as c increases from 0, a new eddy emerges at $x = -3$, an eddy vanishes at $x = 3$ and finally another eddy emerges at $x = -3$. Hence, when $M = 3$, there are either two or three eddies, depending on the value of c. For $M = 4$ and $M = 5$, this pattern of eddies alternately emerging at $x = -M$ and vanishing at $x = M$ as c increases is again observed; there are always either three or four eddies when $M = 4$, and four or five eddies when $M = 5$.

Our results suggest that, for any given ratio of length to width of the boundary ellipse, the number of eddies varies by no more than one eddy as the position of the rotlet varies along the major axis. Furthermore, it appears that, when $1 \leq M \leq M_1$, there are either no eddies or one eddy in the flow, and, when $M_n < M \leq M_{n+1}$, there are either $2n - 1$, $2n$, or $2n + 1$ eddies. Hence, for this particular flow, the boundary geometry seems to be much more important in determining the number of eddies than the position of the rotlet.

Values of ψ and v (the y-component of the flow velocity) at various positions on the major axis $y = 0$ of the flow region were computed using (3.8) and (2.2) for the same values of M and c as were used in Fig. 2. The results appear in Table 2. The velocities given in this table provide a measure of the 'intensity' of the flow at various locations. To illustrate, a measure of the relative intensity of the two eddies to the right of the rotlet in Fig. 2(b) is given by the ratio of the flow speeds at $y = 0$ for the two $\psi = 0$ streamlines to the right of the rotlet. From

Table 2. Values of ψ and v on the axis $y = 0$ for (a) $M = 2$, $c = 0$, (b) $M = 4$, $c = 0$, and (c) $M = 2$, $c = 1.6$. The symbol '*' denotes x values (the first five digits of which are given) where $\psi = 0$. The notation 'En' means multiplication by 10 to the power n.

(a)

x	ψ	v
0.2	−1.264	4.792
0.4	−0.6314	2.107
0.6	−0.3201	1.126
0.8	−0.1512	0.6105
1.0	−0.06183	0.3101
1.2	−0.01886	0.1361
1.4	−0.002008	0.04312
1.4579*	0	0.02692
1.6	0.001837	0.002089
1.8	9.196$E-4$	−0.007374

(b)

x	ψ	v
0.2	−1.275	4.795
0.6	−0.3289	1.139
1.0	−0.06273	0.3352
1.3130*	0	0.09964
1.4	0.007062	0.06427
1.8	0.01424	−0.009676
2.2	0.007937	−0.01659
2.6	0.002764	−0.008869
3.0	5.658$E-4$	−0.002782
3.4	3.292$E-5$	−3.819$E-4$
3.5470*	0	−1.017$E-4$
3.8	−3.181$E-6$	2.556$E-5$

(c)

x	ψ	v
−1.98	−3.850$E-8$	−3.118$E-6$
−1.9294*	0	9.862$E-6$
−1.8	1.126$E-5$	2.077$E-4$
−1.4	5.458$E-4$	0.003206
−1.0	0.003405	0.01244
−0.6	0.01182	0.03149
−0.2	0.03004	0.06094
0.2	0.06022	0.08616
0.6	0.08929	0.03246
1.0	0.03558	−0.4345
1.0674*	0	−0.6320
1.4	−0.5811	−3.941
1.8	−0.4028	4.443

Table 2(b), this relative intensity is seen to be $0.09964/0.0001017 \approx 980$. This value may be compared with the relative intensity of adjacent eddies in the Stokes flow between parallel planar walls caused by a distant two-dimensional disturbance; Moffatt [7], using essentially the same measure of relative intensity as the one used here, found this value to be about 350. (A more accurate value is 358.) These results clearly demonstrate that the intensity of two-dimensional Stokes flows of this kind decreases very rapidly as the distance from the disturbance driving the flow increases.

References

1. B.Y. Ballal and R.S. Rivlin, Flow of a Newtonian fluid between eccentric rotating cylinders: inertial effects. *Arch. Rat. Mech. Anal.* 62 (1976) 237–294.
2. J.R. Blake and A.T. Chwang, Fundamental singularities of viscous flow, Part I: The image systems in the vicinity of a stationary no-slip boundary. *J. Eng. Math.* 8 (1974) 23–29.
3. V.T. Buchwald, Eigenfunctions of plane elastostatics, I. The strip. *Proc. R. Soc. London Ser. A* 277 (1964) 385–400.
4. A.M.J. Davis, M.E. O'Neill, J.M. Dorrepaal and K.B. Ranger, Separation from the surface of two equal spheres in Stokes flow. *J. Fluid Mech.* 77 (1976) 625–644.
5. W.W. Hackborn, Asymmetric Stokes flow between parallel planes due to a rotlet. *J. Fluid Mech.* 218 (1990) 531–546.
6. G.B. Jeffery, The rotation of two circular cylinders in a viscous fluid. *Proc. R. Soc. London Ser. A* 101 (1922) 169–174.
7. H.K. Moffatt, Viscous and resistive eddies near a sharp corner. *J. Fluid Mech.* 18 (1964) 1–18.
8. K.B. Ranger, Eddies in two dimensional Stokes flow. *Int. J. Engng Sci.* 18 (1980) 181–190.
9. G.H. Wannier, A contribution to the hydrodynamics of lubrication. *Q. Appl. Math.* 8 (1950) 1–18.